

# Closed-form expansions for the universal edge elimination polynomial

K. Dohmen

*Department of Mathematics, Mittweida University of Applied Sciences, Germany*  
*Electronic address: dohmen@hs-mittweida.de*

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*Abstract.* We establish closed-form expansions for the universal edge elimination polynomial of paths and cycles and their generating functions. This includes closed-form expansions for the bivariate matching polynomial, the bivariate chromatic polynomial, and the covered components polynomial.

*Keywords.* edge elimination polynomial, bivariate matching polynomial, bivariate chromatic polynomial, covered components polynomial, generating function, path, cycle, closed-form

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## 1 Introduction

As a generalization of several well-known graph polynomials, Averbouch, Godlin and Makowsky [1] introduced the so-called *universal edge elimination polynomial*  $\xi(G, x, y, z)$ , whose recursive definition involves three kinds of edge elimination:

$G_{-e}$ : The graph obtained from  $G$  by removing the edge  $e$ .

$G_{/e}$ : The graph obtained from  $G$  by removing  $e$  and identifying its endpoints,

$G_{\dagger e}$ : The graph obtained from  $G$  by removing  $e$  and all incident vertices.

All graphs are considered as finite and undirected, and may have loops and multiple edges. We use  $P_n$  to denote the simple path with  $n$  vertices ( $n = 0, 1, \dots$ ), and  $\oplus$  to denote the disjoint union of graphs. According to [1],  $\xi(G, x, y, z)$  is defined by

$$\xi(P_0, x, y, z) = 1, \quad \xi(P_1, x, y, z) = x, \quad (1)$$

$$\xi(G, x, y, z) = \xi(G_{-e}, x, y, z) + y\xi(G_{/e}, x, y, z) + z\xi(G_{\dagger e}, x, y, z), \quad (2)$$

$$\xi(G_1 \oplus G_2, x, y, z) = \xi(G_1, x, y, z)\xi(G_2, x, y, z). \quad (3)$$

The universal edge elimination polynomial  $\xi(G, x, y, z)$  generalizes, among others, the bivariate matching polynomial  $M(G, x, y) = \xi(G, x, 0, y)$  (provided  $G$  is loop-free), the bivariate chromatic polynomial  $P(G, x, y) = \xi(G, x, -1, x - y)$ , and the covered components polynomial  $C(G, x, y, z) = \xi(G, x, y, xyz - xy)$ . The implications of our results on  $\xi(G, x, y, z)$  for these polynomials are new as well. We refer to [1–4] for the definitions of the various graph polynomials and the relationships among them.

## 2 Closed-form expansions for paths and cycles

We use  $\mathbb{N}$  to denote the set of positive integers. The following theorem provides a closed-form expansion for the universal edge elimination polynomial of a path.

**Theorem 2.1.** *Let  $n \in \mathbb{N}$ , and  $x, y, z \in \mathbb{R}$ . If  $z > -\left(\frac{x+y}{2}\right)^2$ , then*

$$\xi(P_n, x, y, z) = \frac{\sqrt{D} - x + y}{2\sqrt{D}} \left( \frac{x + y - \sqrt{D}}{2} \right)^n + \frac{\sqrt{D} + x - y}{2\sqrt{D}} \left( \frac{x + y + \sqrt{D}}{2} \right)^n \quad (4)$$

where

$$D := x^2 + 2xy + y^2 + 4z. \quad (5)$$

If  $z < -\left(\frac{x+y}{2}\right)^2$ , then

$$\xi(P_n, x, y, z) = (-z)^{n/2} \left( \cos(n\varphi) + \frac{x - y}{\sqrt{-D}} \sin(n\varphi) \right) \quad (6)$$

where

$$\varphi = \begin{cases} \arctan \frac{\sqrt{-D}}{x+y} & \text{if } x + y > 0, \\ \pi/2 & \text{if } x + y = 0, \\ \pi + \arctan \frac{\sqrt{-D}}{x+y} & \text{if } x + y < 0. \end{cases} \quad (7)$$

If  $z = -\left(\frac{x+y}{2}\right)^2$ , then

$$\xi(P_n, x, y, z) = \frac{(n+1)x - (n-1)y}{2} \left( \frac{x+y}{2} \right)^{n-1}. \quad (8)$$

*Proof.* By choosing  $e$  as an end edge of  $P_n$ , Eqs. (2) and (3) yield the recurrence

$$\xi(P_n, x, y, z) = (x + y)\xi(P_{n-1}, x, y, z) + z\xi(P_{n-2}, x, y, z) \quad (n \geq 2), \quad (9)$$

where the initial conditions are given by Eq. (1). This is a homogeneous linear recurrence of degree 2 with constant coefficients. We solve this recurrence by applying the method of characteristic roots. The characteristic equation of the recurrence is

$$r^2 - (x + y)r - z = 0, \quad (10)$$

with discriminant  $D$ , given by Eq. (5). In our three cases, we have  $D > 0$ ,  $D < 0$ , and  $D = 0$ , respectively. In the first two cases, the solution to Eq. (9) is of the form

$$\xi(P_n, x, y, z) = c_1 r_1^n + c_2 r_2^n \quad (11)$$

where  $r_1, r_2$  are the distinct roots of Eq. (10) and  $c_1, c_2$  are chosen to satisfy Eq. (1). In the first case we have

$$\begin{aligned} r_1 &= \frac{x+y-\sqrt{D}}{2}, & c_1 &= \frac{\sqrt{D}-x+y}{2\sqrt{D}}, \\ r_2 &= \frac{x+y+\sqrt{D}}{2}, & c_2 &= \frac{\sqrt{D}+x-y}{2\sqrt{D}}, \end{aligned} \quad (12)$$

and in the second case,

$$\begin{aligned} r_1 &= \frac{x+y}{2} - \frac{\sqrt{-D}}{2}i, & c_1 &= \frac{1}{2} + \frac{x-y}{2\sqrt{-D}}i, \\ r_2 &= \frac{x+y}{2} + \frac{\sqrt{-D}}{2}i, & c_2 &= \frac{1}{2} - \frac{x-y}{2\sqrt{-D}}i. \end{aligned} \quad (13)$$

A little bit of extra work is needed in the second case in order to get rid of the imaginary parts: Representing  $r_1$  and  $r_2$  in polar form and applying Euler's formula we obtain

$$\begin{aligned} r_1^n &= (\sqrt{-z} e^{-i\varphi})^n = (-z)^{n/2} (\cos(n\varphi) - \sin(n\varphi)i), \\ r_2^n &= (\sqrt{-z} e^{i\varphi})^n = (-z)^{n/2} (\cos(n\varphi) + \sin(n\varphi)i), \end{aligned} \quad (14)$$

with  $\varphi$  as in Eq. (7). Thus, Eq. (11) becomes

$$\begin{aligned} \xi(P_n, x, y, z) &= (-z)^{n/2} \left( \frac{1}{2} \cos(n\varphi) + \frac{x-y}{2\sqrt{-D}} \cos(n\varphi)i - \frac{1}{2} \sin(n\varphi)i + \frac{x-y}{2\sqrt{-D}} \sin(n\varphi) \right. \\ &\quad \left. + \frac{1}{2} \cos(n\varphi) - \frac{x-y}{2\sqrt{-D}} \cos(n\varphi)i + \frac{1}{2} \sin(n\varphi)i + \frac{x-y}{2\sqrt{-D}} \sin(n\varphi) \right). \end{aligned}$$

This shows that the imaginary parts cancel out. This proves Eq. (6).

In the third case, the solution to Eq. (9) is  $\xi(P_n, x, y, z) = (c_1 + c_2 n)r^n$  where  $r = \frac{x+y}{2}$  is the unique root of Eq. (10) and  $c_1, c_2 \in \mathbb{R}$  are determined by Eq. (1). If  $x+y=0$ , then  $\xi(P_n, x, y, z) = 0$ . Thus, in this case, Eq. (8) holds. If  $x+y \neq 0$ , then by Eq. (1),  $c_1 = 1$  and  $c_2 = \frac{x-y}{x+y}$ ; hence,

$$\xi(P_n, x, y, z) = \left( 1 + \frac{x-y}{x+y}n \right) \left( \frac{x+y}{2} \right)^n,$$

which coincides with Eq. (8). This completes the proof.  $\square$

For any  $n \in \mathbb{N}$ , we use  $C_n$  to denote the connected 2-regular graph with  $n$  vertices. We adopt the convention that  $C_0$  is the empty graph. By Eq. (2) we have

$$\xi(C_1, x, y, z) = x + xy + z, \quad (15)$$

$$\xi(C_2, x, y, z) = x^2 + 2xy + 2z + xy^2 + yz. \quad (16)$$

The following theorem generalizes Eqs. (15) and (16) to cycles of any finite length.

**Theorem 2.2.** *Let  $n \in \mathbb{N}$  and  $x, y, z \in \mathbb{R}$ . Let  $D$  and  $\varphi$  be defined as in Eq. (5) resp. (7). If  $z \geq -\left(\frac{x+y}{2}\right)^2$ , then*

$$\xi(C_n, x, y, z) = \left(\frac{x+y-\sqrt{D}}{2}\right)^n + \left(\frac{x+y+\sqrt{D}}{2}\right)^n + y^{n-1}(xy - y + z). \quad (17)$$

If  $z \leq -\left(\frac{x+y}{2}\right)^2$ , then

$$\xi(C_n, x, y, z) = 2(-z)^{n/2} \cos(n\varphi) + y^{n-1}(xy - y + z). \quad (18)$$

*Proof.* For  $n = 1, 2$  the theorem agrees under both conditions on  $z$  with Eqs. (15) and (16). This is easy to see for  $z \geq -\left(\frac{x+y}{2}\right)^2$ , while for  $z \leq -\left(\frac{x+y}{2}\right)^2$  the identities  $\cos(\arctan(t)) = 1/\sqrt{1+t^2}$  and  $\cos(\alpha) = 2\cos^2(\alpha) - 1$  reveal the coincidence.

For the rest of this proof, we assume  $n \geq 3$ . We may further assume that  $z \neq -\left(\frac{x+y}{2}\right)^2$  as the remaining case follows for reasons of continuity by taking limits on both sides of Eqs. (17) and (18) as  $z \downarrow -\left(\frac{x+y}{2}\right)^2$  resp.  $z \uparrow -\left(\frac{x+y}{2}\right)^2$ . By Eq. (2) we have the non-homogeneous recurrence

$$\xi(C_n, x, y, z) = \xi(P_n, x, y, z) + y\xi(C_{n-1}, x, y, z) + z\xi(P_{n-2}, x, y, z) \quad (n \geq 3)$$

with initial condition as in Eq. (16). Iterating this recurrence gives

$$\begin{aligned} \xi(C_n, x, y, z) &= \sum_{j=0}^{n-2} y^j \left( \xi(P_{n-j}, x, y, z) + z\xi(P_{n-j-2}, x, y, z) \right) + y^{n-1}(x + xy + z) \\ &= \xi(P_n, x, y, z) + y\xi(P_{n-1}, x, y, z) + (y^2 + z) \sum_{j=0}^{n-4} y^j \xi(P_{n-j-2}, x, y, z) \\ &\quad + y^{n-3} (xz + yz + xy^2 + xy^3 + y^2z). \end{aligned} \quad (19)$$

Using Eq. (11) with  $c_1, r_1, c_2, r_2$  from Eqs. (12) and (13) in the preceding proof, the sum on the right-hand side of Eq. (19) can be written as

$$\begin{aligned} \sum_{j=0}^{n-4} y^j \xi(P_{n-j-2}, x, y, z) &= \sum_{j=0}^{n-4} y^j (c_1 r_1^{n-j-2} + c_2 r_2^{n-j-2}) \\ &= c_1 r_1^{n-2} \sum_{j=0}^{n-4} \left(\frac{y}{r_1}\right)^j + c_2 r_2^{n-2} \sum_{j=0}^{n-4} \left(\frac{y}{r_2}\right)^j. \end{aligned}$$

If  $z \neq -xy$ , then  $y \neq r_1$  and  $y \neq r_2$ . In this case, by applying the formula for finite geometric series the preceding equation simplifies to

$$\sum_{j=0}^{n-4} y^j \xi(P_{n-j-2}, x, y, z) = c_1 r_1^2 \frac{r_1^{n-3} - y^{n-3}}{r_1 - y} + c_2 r_2^2 \frac{r_2^{n-3} - y^{n-3}}{r_2 - y}.$$

Substituting this latter expression into Eq. (19) and taking into account that  $r_1$  and  $r_2$  are given as in Eqs. (12) and (13) leads to

$$\begin{aligned} \xi(C_n, x, y, z) &= c_1 r_1^n + c_2 r_2^n + y(c_1 r_1^{n-1} + c_2 r_2^{n-1}) \\ &\quad + (y^2 + z) \left( c_1 r_1^2 \frac{r_1^{n-3} - y^{n-3}}{r_1 - y} + c_2 r_2^2 \frac{r_2^{n-3} - y^{n-3}}{r_2 - y} \right) \\ &\quad + y^{n-3} (xz + yz + xy^2 + xy^3 + y^2 z) \\ &= r_1^n + r_2^n + y^{n-1} (xy - y + z), \end{aligned} \tag{20}$$

where the last equality follows by substituting  $c_1 = -\frac{r_1 - y}{\sqrt{D}}$ ,  $c_2 = \frac{r_2 - y}{\sqrt{D}}$ ,  $\sqrt{D} = -\frac{r_1^2 - r_2^2}{x + y}$ , and rearranging and cancelling terms (note that  $\sqrt{D} = i\sqrt{-D}$  if  $D < 0$ ). Now, for  $z > -(\frac{x+y}{2})^2$  Eq. (17) follows from Eqs. (20) and (12), whereas for  $z < -(\frac{x+y}{2})^2$  Eq. (18) follows from Eqs. (20) and (14) after cancelling out the imaginary parts, in analogy to the proof of Theorem 2.1.

If  $z = -xy$ , then  $z > -(\frac{x+y}{2})^2$ . In this remaining case, the result follows for reasons of continuity by taking limits on both sides of Eq. (17) as  $z \downarrow -xy$ .  $\square$

*Remark 2.3.* For  $z = -(\frac{x+y}{2})^2$ , Eqs. (17) and (18) coincide. In this case,

$$\xi(C_n, x, y, z) = 2 \left( \frac{x + y}{2} \right)^n - \frac{x^2 - 2xy + y^2 + 4y}{4} y^{n-1}.$$

Alternatively, this can be shown by combining Eqs. (8) and (19) and applying the formula for finite geometric series.

*Remark 2.4.* The preceding closed-form expansions can also be proved by induction. A computer algebra system might be helpful. In **Sage** [5], for instance, the following lines of code prove Eqs. (4) and (17) by induction on the number of vertices.

```
var("n x y z")
D = x^2+2*x*y+y^2+4*z
path = (sqrt(D)-x+y)/(2*sqrt(D))*((x+y-sqrt(D))/2)^n \
      +(sqrt(D)+x-y)/(2*sqrt(D))*((x+y+sqrt(D))/2)^n
cycle = ((x+y-sqrt(D))/2)^n+((x+y+sqrt(D))/2)^n+y^(n-1)*(x*y-y+z)
bool(path(n=0)==1 and path(n=1)==x \
      and (x+y)*path(n=n-1)+z*path(n=n-2)==path)
bool(cycle(n=1)==x*x*y+z and path+y*cycle(n=n-1)+z*path(n=n-2)==cycle)
```

We proceed with a corollary on the generating function of  $\xi(G, x, y, z)$ .

**Corollary 2.5.**

$$\sum_{n=0}^{\infty} \xi(P_n, x, y, z) t^n = \frac{1 - yt}{1 - (x + y)t - zt^2},$$
$$\sum_{n=0}^{\infty} \xi(C_n, x, y, z) t^n = \frac{1 + zt^2}{1 - (x + y)t - zt^2} + \frac{(xy - y + z)t}{1 - yt}.$$

*Proof.* Corollary 2.5 is an immediate consequence of Theorem 2.1, Theorem 2.2 and the geometric series formula.  $\square$

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## References

- [1] I. Averbouch, B. Godlin and J.A. Makowsky, *An extension of the bivariate chromatic polynomial*, Europ. J. Combin. **31** (2010), 1–17.
- [2] K. Dohmen, A. Pönitz, and P. Tittmann, *A new two-variable generalization of the chromatic polynomial*, Discrete Math. Theoret. Comput. Sci. **6** (2003), 69–90.
- [3] I. Gutman and F. Harary, *Generalizations of the matching polynomial*, Util. Math. **24** (1983), 97–106.
- [4] M. Trinks, *The covered components polynomial: A new representation of the edge elimination polynomial*, Electron. J. Combin. **19** (2012), #P50, 31 pp.
- [5] W.A. Stein et al., *Sage Mathematics Software (Version 6.3)*, The Sage Development Team, 2014, <http://www.sagemath.org>.